Challenges and Constraints for Constructing Curriculum

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A keynote address
at the 17th Annual Conference
of the Mathematics Education Research Group of Australasia

Southern Cross University
Lismore
Australia 2480
5-8 July 1994

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This paper describes some of the broad lessons on curriculum which have emerged from "The Cognitive and Linguistic Demands of Learning To Use Algebra", a research project focussing on students' understandings of algebraic notation. Much of the current curriculum in number and algebra is designed principally to support algorithm development, whether skills or concept based. As technology increasingly frees us from an algorithm-driven curriculum, this should be replaced by experiences which expand children's conceptions of numbers and the operations of them. To learn algebra, students need experiences which assist them to become alert to multiple meanings and to explicitly recognise processes. Teachers need to address students' expectations of notation systems.

The importance of building a curriculum on what children know is widely recognised. Children construct new knowledge and patterns of thinking when new experiences interact with prior knowledge and thinking. A major challenge for mathematics education today is to build into curriculum practice the insights into children's ways of thinking that have emerged from research over the past two decades. In this paper, some of the broad lessons on curriculum which have emerged from the research project "The Cognitive and Linguistic Demands of Learning To Use Algebra" will be described. This work has been carried out jointly with Dr Mollie MacGregor and was funded by the Australian Research Council. The research focussed on students' understanding and use of algebraic notation. It highlights the need for students to develop a much deeper understanding of the properties of numbers and their operations than is generally the case.

The first part of the paper outlines how algebra depends on thinking that is alert to multiple meanings and where processes are explicitly recognised. Then I report some of the expectations that students have of algebraic notation that it does not meet, because of the very restricted set of ideas that it can represent. A further challenge to curriculum construction is the mismatch between many students' concepts of number operations and the broad conceptions required for algebra. Finally, some new emphases for curriculum are briefly mentioned.

DEVELOPING UNCOMMON SENSE

To do well in mathematics, students need to have a strong everyday common sense about their work. They need to feel an intuitive understanding about number and space and the effects of operating on them. A lot of the work of school mathematics, particularly as emphasised in recent years, is properly directed to this end. However, in learning algebraic notation, students have to be able to step beyond the realms of everyday, intuitive thinking and deal with a formal system. Balancing the emphasis in school mathematics of these two types of thinking is a central challenge for school mathematics.

Intuitive thinking is characterised by Fischbein (1987) as immediate cognition, where the sense is obtained from the elements and context involved. The meaning of a statement can often be constructed by common sense and knowing what the elements in the sentence are. There is an immediate, intuitive sense, for example, to a statement containing the words "dog", "man", "bite", "hospital". In most intuitive arithmetic operations, there is some direct meaning available at every stage, so for example, it is easier to understand division by repeated subtraction where at any stage you can say "I had 300 and now I have taken away four lots of 50 and more 50's can still be taken away" than it is to understand long division. A further characteristic of intuitive thinking, possibly related to its immediacy, is that mental procedures are often carried out without the person consciously being able to describe what has been done. This is a phenomenon well-documented in a variety of settings (e.g. Gurova, 1969 and also see below), that is easy to overlook when interviewing children or trying to analyse written reports of thinking.

In contrast, for using algebraic notation students' thinking needs to be quite different. They very often have to be alert to the possibilities of multiple meanings, without obvious clues from context to prompt them. The meaning of a statement often depends on the syntax rather on the elements and there may be no context to guide. Whereas in primary school arithmetic students can usually be guided intuitively by the size of the numbers involved and the sense of what the answer would be when using algebra, students need to be explicit about the procedures they are to carry out, not about the answers they get. Algebraic notation is centrally concerned with describing procedures. Finally, although there must be an overall sense to the algebraic procedure, there may be intermediate steps (e.g. in equation solving) which cannot readily be translated back in a meaningful way to the problem situation from which it arose. Some contrasts between the two modes of thinking are illustrated in the examples below. The data is drawn from a series of written tests and interviews involving large numbers of students in Years 7 to 11, from a variety of schools across socio-economic groups principally in Victoria.

Firstly we have had in our tests several items which have looked at reading and comprehension of quantitative English with simple vocabulary. Sample problems were to find the dose for a child from a medicine label and to select flights and trains from timetables. In these tasks, we were able to conclude that very few students (perhaps no more than 2% at Year 10 across a variety of schools) have problems in identifying and using information located in forms and tables. These results were markedly better than those of a large national survey of adult literacy on similar questions (MacGregor & Stacey, 1994; Wickert, 1989).

However, when attention needs to be given to word order or syntax many students made mistakes. For example, item MATCH gives a drawing of a match and underneath gives the instruction "The match is 2 cm shorter than the line. Draw the line". An appropriate space is indicated for drawing the line. This item is consistently badly done. Many students draw a line 2 cm shorter than the match, not vice versa. However, when this item appeared as the last in a series of other similar items (e.g. "Draw a line 1 cm shorter than the match".), students were prompted to think about the possible multiple meanings involved and read with careful attention to the syntax. Many more students were correct. A similar result has been observed with a standard algebra item. Success rates for students choosing an equation to correspond with the statements:

> "In a class there are six more boys than girls. If we write p for the number of girls and s for the number of boys, which of the following equations are correct?"

were considerably improved when the item was preceded by a question which asked whether it was appropriate to add the 6 onto the number of boys or girls. In this case, we contend that the mental model that students form intuitively (without the prompt) for comparison items such as these, leads students to write the reversed equation (MacGregor & Stacey, 1993a), whereas the prompting forces more students to an examination of the multiple possibilities and a conscious choice to be made. It has been found that students' performance on reading items which require attention to syntax and word order puts a ceiling on their performance in algebra (MacGregor & Stacey, in press).

Another instance where openness to multiple meanings may be involved is in the use of brackets. Items where students had to put in brackets were consistently very poorly done at all schools. However a short teaching intervention was very effective. It focussed on the possible alternative meanings with and without brackets and raised success rates in one school from 25% before the teaching to 70% six months later. The success rate later dropped back to 60% , indicating that practice is important.

THE NEED FOR EXPLICIT AWARENESS OF OPERATIONS

Children with a good intuitive sense of number can often find correct answers without being able to say what they have done. It is not that students don't have the vocabulary or grammar, but probably that the mental processing is not occurring in a verbal way. This has immediate consequences for learning algebra, where the notation is powerful because it expresses operations explicitly in both general statements and general methods. Several of our test items have asked students to express in words and symbols patterns in tables of integer values of linear functions. Even amongst the students who showed that they "knew" what the relationships were by calculating difficult numerical values, a quarter could not describe the rules they used verbally and nearly a half were not able to describe them algebraically (MacGregor & Stacey, 1993b).

Equally striking was data which compared students' ability to use algebraic notation with their understanding of mathematical relationships expressed graphically. In one of our tests given twice to 22 Year 10 students, most were able to pick a graph which showed the relationship between x and y corresponding to " I buy x bags of peanuts costing y cents each and the total cost is 600 cents" yet only 20% could write any correct equation linking x and y. Students understand far more than they can say, but being able to read and write algebraic notation will remain important because it will be an essential link into using new mathematical software tools.

WHAT ALGEBRA CAN AND CANNOT SAY

Students draw upon their previous experiences with a variety of different formal and informal codes to interpret and write algebraic expressions. We, and others, have observed that when they write algebraic expressions, students draw analogies with alphabetic codes (so that H is 8, H + 10 is 18 or R and y - 1 is x), with codes where juxtaposition (conjoining) means addition (so 3x is 3+x in the same way as $3\frac{3}{4}$ is $3 + \frac{3}{4}$), with the writing of shorthand notes (so that letters are thought to stand for words) and with units of measurement (so that in a question where the variable is denoted by x, an x is added to all the answers as a sort of unit). These and other misunderstandings of the meaning of letters are extensively discussed in the research literature, as is the limited understanding of the equals sign as a signal simply to work something out. However, our work show that the students' misunderstandings of algebraic notation are much more pervasive, involving the operations as well and an expectation that all mathematical ideas can be written in algebra, just as all ideas can be indicated in a written note.

Many students use a symbol system which looks superficially like algebra but in fact has quite different meanings, as a sort of personal shorthand to record ideas in an ad hoc way. I first want to stress the very ad hoc nature of many students' writing of algebra. Although they have derived their ideas from written codes that they have previously experienced, we have found very few instances of students consistently using non-standard interpretations. Instead they swap between codes, responding to visual and other clues - would a "y" go in that position in an equation?, do answers usually look like this? In one study (Stacey & MacGregor, in press), we looked at the incidence of the conjoining error (where students write ab instead of a+b) in various tasks. The conjoining error is believed by some researchers to be highly significant because of its relationship to "acceptance of lack of closure", an indicator of cognitive growth. We found that hardly any students used conjoining for addition consistently and for addition alone. There was considerable variation in the incidence of conjoining errors from question to question, ranging from none at all in formulating a simple equation to 3% for simple equation solving to about 15% when Year 10 students transposed a difficult equation. Students do not apply consistent but wrong laws to write algebraic notation. Instead, many simply seemed to put relevant numbers and symbols together on the page in a way that superficially looked like the algebra they were used to seeing.

The possibility that a symbol system that looks superficially like algebra being mistaken for algebra was brought home to me when I was told of a Year 11 Chemistry student who was puzzled by the formula $Ca(NO_3)_2$. He asked his teacher "do you clear the brackets first or multiply by the calcium"? Then I read of Antoine Lavoisier, the chemist who showed that combustion was better explained by oxygen than by phlogiston. Before he was guillotined in the French Revolution, he established the modern system of naming compounds after their constituent elements and also popularised the use of formulae and equations in chemistry. In introducing these, he said "I have constituted formulae of a kind that could at first be taken for algebraic formulae, but do not have the same object.....I beg you to consider the formulae [..] only as simple annotations, the object of which is to ease the workings of the mind" (p35). Lavoisier was worried about the possibility that a system which looked superficially like algebra would be misinterpreted as algebra. The objects (atoms, not numbers) and the meaning of the operations (conjoining meaning "already chemically combined" and "+" meaning "being chemically combined") are all quite different from the meanings of algebra although the symbols look very similar, as the Year 11 student recognised.

One of the difficulties for teaching algebra is that it is hard to explain how things that look the same can have quite different meanings and it is also hard for a teacher to detect when a student writes a symbolic statement with an irregular interpretation. When students in Year 10 were asked to write algebraically a relationship between x and y, given the information in Figure 1, the success rates were very low. Some of the ways in which students tried to say that y is equal to x+4 were:

x=y4 (said by the student to say "with the x, for the y you put 4 on it"),

x = 1 + 4y (you start with x equal to 1 and add 4 to get y)

x+4y (you take x and add 4 to get y)

x=3y ("x equals three digits y" - there are 3 integers between an x and its y value).

Other students (and the same students at other times) saw the most salient features of the table to be hat the x values increase by 1 and the y values increase by 1 (but start at a different number). They wrote expressions such as x+1 = y + 1 (every time you add one to x you have to add one to y) and 1x = 5y (x starts at 1 and y starts at 5). These students have perceived and have tried to express patterns algebraically but do not realise what a restricted system elementary school algebra is. As teachers we become familiar with the ways of getting around the restrictions and so stop seeing them. We promote algebra as the language to describe patterns. We probably also stop even noticing the patterns in the tables that we cannot express algebraically.

One of the most popular approaches to algebra, and the one which is endorsed in the *National Statement* (Australian Education Council, 1991), is to offer students tables of integer values (such as in Figure 1) or sequences of geometric designs and get them to write an algebraic statement of the relationships. Unfortunately the most salient "successor" features of the table in Figure 1 cannot be written in the notation the students are learning. Instead students write expressions which superficially look algebraic but are meant to convey very different meanings as the examples above show. One teaching strategy that may help here is to use frequently examples of linear relationships which are not presented so that the "successor" relationship stands out. Many everyday instances of linear functions (cost of petrol purchase as a function of number of litres bought, telephone bill as a function of number of calls) are suitable to use because they do not naturally arise in a sequence. However, the more general point is that students have to learn, and teachers have to see afresh, how few of the patterns and relationships that are observed in a mathematical situation can be directly written algebraically.

x	1	2	3	4	5	6	7	8			
y	5	6	7	8	9	10	11	12			

Figure 1: Table illustrating relationship to be written algebraically.

THE NEED FOR BROADER CONCEPTS OF OPERATIONS

One of the obstacles to students describing verbally or algebraically the rules that they use in calculation is that they do not have a well integrated understanding of the four arithmetic operations in all their various guises. At the most simple level, some of the Year 10 students interviewed used descriptions based on counting rather than on addition, such as "there's three numbers missing" and "in between x and y there is four" to describe y = x+4 (MacGregor & Stacey, 1993b).

In arithmetic, especially with whole numbers, there are often many possible paths for working out problems. For example solutions to a question such as SCHOOL FETE (Figure 2) from a typical class of students beginning algebra would probably show all variations given in the Figure. In contrast, there are no significant variations on the algebraic solution to a parallel task. If the class brings D items per day to reach a target of T items, then it will take T/(5D) weeks. Teachers therefore need to work with beginning algebra students to expand their ability to recognise the applicability of operations (especially multiplication and division) in a wide range of cognitively and structurally different problems. Conversely, children need to learn to see a unity amongst the various ways in which operations are carried out. It is important that they see that subtraction of whole numbers, fractions and decimals share the same properties although they are carried out differently with pencil and paper.

Within the context of whole numbers, the only incentive to move towards using a more sophisticated method (e.g. multiplication or division instead of repeated addition) is overall ease of solving the problem. A child will only perceive a method to be easy if the method of calculation itself is felt to be easy and reliable. For this reason, regular use of calculators could be very beneficial, as it can help focus children's attention on the easiest operation to use, not the easiest way to actually do the calculation. Some evidence from the *Calculators in Primary Mathematics* project (a joint project of Deakin and Melbourne Universities) supports this proposal. Grades 3 and 4 children who had used calculators throughout their schooling were better able to choose an appropriate operation in a word problem than other children from the same schools who had not regularly used calculators (Stacey & Groves, 1994).

SCHOOL FETE

Our class is collecting items for the white elephant stall at the school fete. If we collect an average of ten items per day, how many weeks will it take us to collect 500 items?

Trial Addition

```
10          50
10          50
10          50
10          50
10          50
--
50 each     50
   week     50
            50
            50
            50
            --
           500
Answer = 10 weeks
```

Repeated Subtraction

```
 500        250
 -50        -50
 ---        ---
 450        200
 -50        -50
 ---        ---
 400        150
 -50        -50
 ---        ---
 350        100
 -50        -50
 ---        ---
 300         50
 -50        -50
 ---        ---
 250          0
```

Division

10 items each day
is 50 items each week

```
      10
  50  500
```

Answer = 10 weeks

Division

```
     50
10 ) 500
```

50 days
= 10 weeks

Trial Multiplication

10 x 5 = 50
50 x 10 = 500

Figure 2. Different ways of solving one problem.

TWO FURTHER SUGGESTIONS FOR TEACHING AND CURRICULUM

(i) Increase the intellectual content of beginning algebra examples.

We have seen algebra worksheets for Year 8 that could easily be done by children in Grade 1 if x was replaced by a place holding box. We have watched teachers teach equation solving using equations that are trivial to guess (and could be guessed by Grade 1 children). In fact, it is years before some students meet an algebraic equation that cannot be easily guessed. This is partly because of the total ban on numbers other than positive whole numbers and partly because of the total ban on looking at any equations not of the form $ax+b = c$. The use of examples like these sends strong messages to students that mathematics is useless mumbo jumbo, a complicated way of doing things that are easy. Student learn to stop watching their teachers and then when they need algebraic methods (e.g for an equation of the form ax+b = cx + d) they haven't learned anything.

(ii) Re-examine the purpose of concept development in number.

Particularly in primary schools, teachers generally give good attention to the need for children to build abstract ideas on concrete experiences. However, the main thrust of this concept development is towards preparing for written algorithms. Decisions not to do algorithms (for example, for division of fractions and decimals in primary schools) have meant in practice that nothing is done about extending concepts of division to incorporate numbers that are not whole. As technology increasingly frees us from an algorithm-driven curriculum, an important challenge for mathematics educators is to create curriculum experiences which help students construct deep conceptions of how numbers and operations work and how they interact with each other. Students' success in this area will determine the extent to which they are able to tap into the power that algebra as a symbolic system, and through it a range of new mathematical tools, can provide.

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